(1)

On Statman's Finite Completeness Theorem

Richard Statman Gilles Dowek

June 1, 1992
CMU-CS-92-152

School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213-3890

Abstract

We give a complete self-contained proof of *Statman's finite completeness theorem* and of a corollary of this theorem stating that the λ-definability conjecture implies the higher-order matching conjecture.

Sponsored by the Avionics Laboratory, Wright Research and Development Center, Aeronautical Systems Division (AFSC), U.S. Air Force, Wright-Patterson AFB, Ohio 45433-6543 under Contract F33615-90-C-1465, ARPA Order No. 7597.

Keywords: simply typed lambda-calculus, finite model, lambda-definability, higher-order matching

The purpose of this note is to give a complete self-contained proof of *Statman's finite completeness theorem* and of a corollary of this theorem stating that the λ-definability conjecture implies the higher-order matching conjecture. Both results are proved in [8] (theorem 2 and 5). Although, since [8] assumes some familiarity with typed λ-calculus model theory and presents several results in short space, it may be not very accessible to readers not familiar with this subject.

Section 1 gives the basic notations used in the paper. The reader not familiar with simply typed λ-calculus should consult Hindley and Seldin [3]. Section 2 presents standard models for simply typed λ-calculus, it is based on Henkin [2]. Section 3 presents the Completeness theorem, it is based on Friedman [1], Plotkin [5] [6] and Statman [9]. Section 4 presents the construction of a model for some equational theories. Section 5 presents Statman's finite completeness theorem. Both section 4 and 5 are based on [8]. Section 6 presents the λ-definability conjecture. The notion of λ-definability is taken from Plotkin [5] [6]. This conjecture has been studied by Plotkin and Statman. At last section 7 presents the higher order matching conjecture and the proof that the λ-definability conjecture implies the higher order matching conjecture. The decidability of higher order matching is conjectured in Huet [4]. the equivalence of the higher order matching problem and the higher order matching problem with closed terms if proved in [7] and the proof that the λ-definability conjecture implies the higher order matching conjecture is from [8], this proof is also discussed in Wolfram [10].

1 Typed λ-calculus

1.1 Types

The set of types is defined by

$$T = \iota \mid (T \to T)$$

The notation $\alpha \to \beta \to \gamma$ is an abbreviation for $(\alpha \to (\beta \to \gamma))$.
Obviously a type can be written in a unique way $\alpha = \beta_1 \to \dots \to \beta_n \to \iota$.
If α is a type, the *order* of α ($o(\alpha)$) is inductively defined by

$$o(\iota) = 1$$

and

$$o(\alpha \to \beta) = max\{1 + o(\alpha), o(\beta)\}$$

1.2 Terms

A *context* is a set of pairs $\Gamma = \{< x_i, \alpha_i >\}$ where x_i is a variable and α_i is a type, such that if $< x, \alpha > \in \Gamma$ and $< x, \beta > \in \Gamma$ then $\alpha = \beta$.
The set of terms is defined by

$$t = x \mid (t\ t) \mid \lambda x : T.t$$

The notation $(t\ u\ v)$ is an abbreviation for $((t\ u)\ v)$.
The judgement *the term t has type α in the context* Γ $(\Gamma \vdash t : \alpha)$ is inductively defined by

- if $< x, \alpha > \in \Gamma$ then $\Gamma \vdash x : \alpha$,
- if $\Gamma \vdash t : \alpha \to \beta$ and $\Gamma \vdash u : \alpha$ then $\Gamma \vdash (t\ u) : \beta$,
- if $\Gamma \cup \{x : \alpha\} \vdash t : \beta$ then $\Gamma \vdash \lambda x : \alpha.t : \alpha \to \beta$.

Obviously if a term t has type α and β in a context Γ then $\alpha = \beta$.
We write Λ_α^Γ for the set of terms t such that $\Gamma \vdash t : \alpha$.

1.3 Normalization

If t and u are terms, we write $t[x \leftarrow u]$ for the term obtained by substituting the free occurrences of x by u. We write $t \rhd u$ when t, $\beta\eta$-reduces in some steps to u. A term is said to be $\beta\eta$-normal if it does not contain any redexes. We write $t =_{\beta\eta} u$ when t and u are $\beta\eta$-equivalent. As proved in [3] the reduction relation is strongly normalizable and confluent on well-typed terms, thus a well-typed term has a unique $\beta\eta$-normal form.
Obviously a normal term can be written in a unique way

$$t = \lambda x_1 : \alpha_1 \ldots . \lambda x_n : \alpha_n.(x\ c_1\ \ldots\ c_p)$$

where x is a variable.
Let

$$t = \lambda x_1 : \alpha_1 \ldots . \lambda x_n : \alpha_n.(x\ c_1\ \ldots\ c_p)$$

a normal term of type $\alpha_1 \to \ldots \to \alpha_m \to \iota$. The normal η-long form of t is inductively defined as

$$t' = \lambda x_1 : \alpha_1 \ldots . \lambda x_n : \alpha_n . \lambda x_{n+1} : \alpha_{n+1} \ldots . \lambda x_m : \alpha_m . (x\ c'_1\ \ldots\ c'_p\ x'_{n+1}\ \ldots\ x'_m)$$

where c'_i is the normal η-long form of c_i and x'_j is the normal η-long form of x_j.

2 Standard Models

In typed λ-calculus model theory, we do not look at λ-terms as at functions but rather as at *notations* for set-theoretical functions.

Definition: Standard Model
A *standard model* is a family of sets indexed by types $(M_\alpha)_\alpha$ such that $M_{\alpha \to \beta} = M_\beta^{M_\alpha}$.

Definition: Assignment
Let Γ be a context $(M_\alpha)_\alpha$ be a standard model, an *assignment from* Γ *onto* $(M_\alpha)_\alpha$ is a function ν which maps every variable x of type α of Γ to an element of M_α.

Definition: Interpretation
Let Γ be a context, $(M_\alpha)_\alpha$ be a standard model and ν an assignment from this context onto this model, we define the function $\tilde{\nu}$ which maps every term of Λ_α^Γ to an element of M_α by

- $\tilde{\nu}(x) = \nu(x)$,
- if u is a term of type $\alpha \to \beta$ and v of type α then $\tilde{\nu}(u\ v) = \tilde{\nu}(u)(\tilde{\nu}(v))$,
- if u is a term of type β then for all $d \in M_\alpha$, $\tilde{\nu}(\lambda x : \alpha . u)(d) = \tilde{\nu}^+(u)$ where $\nu^+(x) = d$ and $\nu^+(y) = \nu(y)$ for $y \neq x$.

Notation: Let $(M_\alpha)_\alpha$ be a standard model and t and u two terms with the same type α. we write $(M_\alpha)_\alpha \models t = u$ if for each assignment ν, we have $\tilde{\nu}(t) = \tilde{\nu}(u)$.

Remark: Given a set M, there exists only one standard model, $(M_\alpha)_\alpha$ such that $M_\iota = M$.

Moreover if M and M' are two sets with the same cardinal, then we can construct an obvious isomorphism between the standard models $(M_\alpha)_\alpha$ and $(M'_\alpha)_\alpha$ based on M and M'. Indeed let us consider a bijection Φ between M and M' we construct a family of bijections Φ_α between M_α and M'_α by $\Phi_\iota = \Phi$ and $\Phi_{\alpha\to\beta}(f) = \Phi_\beta \circ f \circ \Phi_\alpha^{-1}$.

Obviously if ν is an assignment onto $(M_\alpha)_\alpha$ then the function ν' which maps every variable x of type α to the element $\Phi_\alpha(\nu(x))$ is an assignment onto $(M'_\alpha)_\alpha$ and for each every t of type α $\tilde{\nu'}(t) = \Phi_\alpha(\tilde{\nu}(t))$. So if t and u are two terms of the same type then $(M_\alpha)_\alpha \models t = u$ if and only if $(M'_\alpha)_\alpha \models t = u$.

So given a cardinal ξ there exists only one standard model (up to isomorphism) such that M_ι has cardinal ξ, we write it M_ξ.

3 Completeness Theorem

Definition: Friedman-Plotkin Model

Assume Γ contains an infinite number of variables of each type. Let $M_\iota = \Lambda_\iota^\Gamma / =_{\beta\eta}$ and $(M_\alpha)_\alpha$ the standard model built on this set.

Proposition: There exists an assignment ν such that for every t of type ι we have $\tilde{\nu}(t) = t/ =_{\beta\eta}$.

Proof: We construct the assignment ν by induction over the order of the types of the variables of Γ. If x has type ι then $\nu(x) = x/ =_{\beta\eta}$. Then assume the definition of $\nu(x)$ given for all the variables x of order strictly lower to k. Let $\beta = \gamma_1 \to ... \to \gamma_n \to \iota$ be a type of order lower or equal to k, t a term of type β and $d \in M_\beta$, we write $t \mathrel{\dot\in} d$ if for all variables $x_1 : \gamma_1, ..., x_n : \gamma_n$ which do not occur free in t we have $(t\ x_1\ ...\ x_n) \in d(\nu(x_1))...(\nu(x_n))$. Obviously if $t \mathrel{\dot\in} d$ and $u \mathrel{\dot\in} d$ then $t =_{\beta\eta} u$.

Let x be a variable of type $\alpha = \beta_1 \to ... \to \beta_n \to \iota$ of order k, we define $\nu(x)$ by

$$\nu(x)(d_1)...(d_n) = (x\ t_1\ ...\ t_n)/ =_{\beta\eta}$$

if there exists $t_1 \mathrel{\dot\in} d_1$, ..., $t_n \mathrel{\dot\in} d_n$ (obviously the element $(x\ t_1\ ...\ t_n)/ =_{\beta\eta}$ does not depend of the choice of $t_1, ..., t_n$) and $\nu(x)(d_1)...(d_n)$ be anything otherwise.

We prove by induction on the structure of the normal η-long form of t that $t \mathrel{\dot\in} \tilde{\nu}(t)$. Let t be a term of type $\beta_1 \to ... \to \beta_n \to \iota$. Since there is in Γ an infinite number of variables of each type there are in Γ variables $x_1 : \beta_1$,

.... $x_n : \beta_n$ which do not occur free in t. Modulo bound variable renaming the term t can be written

$$t = \lambda x_1 : \beta_1 .... \lambda x_n : \beta_n .(x\ u_1\ ...\ u_p)$$

By induction hypothesis, for every i, we have

$$u_i \mathrel{\tilde{\in}} \tilde{\nu}(u_i)$$

so by definition of ν we have

$$\nu(x)(\tilde{\nu}(u_1))...(\tilde{\nu}(u_p)) = (x\ u_1\ ...\ u_n)/=_{\beta\eta}$$

i.e.

$$(x\ u_1\ ...\ u_n) \in \nu(x)(\tilde{\nu}(u_1))...(\tilde{\nu}(u_p))$$

i.e.

$$(x\ u_1\ ...\ u_n) \in \tilde{\nu}(x\ u_1\ ...\ u_p)$$

so

$$(\lambda x_1 : \beta_1 .... \lambda x_n : \beta_n .(x\ u_1\ ...\ u_p)\ x_1\ ...\ x_n) \\ \in \tilde{\nu}(\lambda x_1 : \beta_1 .... \lambda x_n : \beta_n .(x\ u_1\ ...\ u_p))(\nu(x_1))...(\nu(x_n))$$

i.e.

$$(t\ x_1\ ...\ x_n) \in \tilde{\nu}(t)(\nu(x_1))...(\nu(x_n))$$

so

$$t \mathrel{\tilde{\in}} \tilde{\nu}(t)$$

So if t has type ι then $t \in \tilde{\nu}(t)$, i.e. $\tilde{\nu}(t) = t/=_{\beta\eta}$.

Theorem: (Friedman-Plotkin) Completeness Theorem

If t and u are terms of type α then $(M_\alpha)_\alpha \models t = u$ if and only if $t =_{\beta\eta} u$.

Proof: Obviously, if $t =_{\beta\eta} u$ then $(M_\alpha)_\alpha \models t = u$. Conversely, if we have $(M_\alpha)_\alpha \models t = u$ then let us write the type of t and u $\alpha = \beta_1 \to ... \to \beta_n \to \iota$. Since there is in Γ an infinite number of variables of each type, there are in Γ variables $x_1 : \beta_1, ..., x_n : \beta_n$ which do not occur free in t and u. We have

$$(M_\alpha)_\alpha \models (t\ x_1\ ...\ x_n) = (u\ x_1\ ...\ x_n)$$

so using the previous proposition

$$(t\ x_1\ ...\ x_n)/=_{\beta\eta} = (u\ x_1\ ...\ x_n)/=_{\beta\eta}$$

i.e.

$$(t\ x_1\ ...\ x_n) =_{\beta\eta} (u\ x_1\ ...\ x_n)$$

thus

$$t =_{\beta\eta} u$$

Corollary: Let t and u be two terms of type α, $t =_{\beta\eta} u$ if and only if for all standard models M, $M \models t = u$.

Corollary: If ξ is an infinite cardinal and t and u are terms of of type α then $M_\xi \models t = u$ if and only if $t =_{\beta\eta} u$.
Proof: Let us consider a context Γ that contains ξ variables in the type ι and a denumerable number of variables in the other types. The model M constructed above is such that $M \models t = u$ if and only if $t =_{\beta\eta} u$. This model is isomorphic to the model M_ξ so $M_\xi \models t = u$ if and only if $t =_{\beta\eta} u$.

Remark: If n in a finite cardinal then the completeness theorem fails for the model M_n because if M_ι is finite then M_α is finite for every type α and if $(M_\alpha)_\alpha$ verifies the completeness theorem then $M_{\iota\to(\iota\to\iota)\to\iota}$ is infinite. Indeed call

$$\overline{p} = \lambda x : \iota.\lambda f : \iota \to \iota.(f ...(f\ x)...)$$

if p and q are distinct integers then $\overline{p} \neq_{\beta\eta} \overline{q}$ and so $\tilde{\nu}(\overline{p}) \neq \tilde{\nu}(\overline{q})$ and $M_{\iota\to(\iota\to\iota)\to\iota}$ is infinite.

So M_ξ verifies the completeness theorem if and only if ξ is infinite.

4 Equational Theories

Let Γ be a context such that all the types of the variables in Γ are of order at most two. Let E be a set of equations $\{a_i = b_i\}$ where a_i and b_i are well-typed in Γ and have type ι in Γ. We consider the smallest equivalence relation compatible with term structure that contains $=_{\beta\eta}$ and the equations of E. We write this relation $=_{\beta\eta E}$.

Definition: The relation $=_{\beta\eta E}$

The relation $=_{\beta\eta E}$ is inductivelly defined by:

- if $t =_{\beta\eta} u$ then $t =_{\beta\eta E} u$,
- if $a = b \in E$ and c is a term of type $\iota \to \alpha$ then $(c\ a) =_{\beta\eta E} (c\ b)$,

- if $t =_{\beta\eta E} u$ then $u =_{\beta\eta E} t$,
- if $t =_{\beta\eta E} u$ and $u =_{\beta\eta E} v$ then $t =_{\beta\eta E} v$,

Definition: Statman's Model
Let $M_\iota = \Lambda_\iota^\Gamma / =_{\beta\eta E}$. Let $(M_\alpha)_\alpha$ be the standard model built on M_ι.

Proposition: If $(M_\alpha)_\alpha \models t = u$ then $t =_{\beta\eta E} u$.
Proof: Let the assignment ν be defined defined as

- if x has type ι, then $\nu(x) = x/ =_{\beta\eta E}$,
- if x has type $\iota \to ... \to \iota \to \iota$ then $\nu(x) = f$ where

$$f(d_1)...(d_n) = (x\ u_1\ ...\ u_n)/ =_{\beta\eta E}$$

with $u_1 \in d_1$, ..., $u_n \in d_n$ (obviously the class of the term $(x\ u_1\ ...\ u_n)$ does not depend on the choice of $u_1, ..., u_n$).

By induction over the structure of the normal form of t, we have for every term t of type ι, $\tilde{\nu}(t) = t/ =_{\beta\eta E}$. Indeed, because t has type ι its normal form can be written $t = (x\ u_1\ ...\ u_n)$ since t is well-typed in Γ the variable x is of order at most two and the u_i have type ι. So $\tilde{\nu}(u_1) = u_1/ =_{\beta\eta E}$, ..., $\tilde{\nu}(u_n) = u_n/ =_{\beta\eta E}$. And by definition of ν, $\nu(x)$ maps $u_1/ =_{\beta\eta E}$, ..., $u_n/ =_{\beta\eta E}$ to $(x\ u_1\ ...\ u_n)/ =_{\beta\eta E}$. So $\tilde{\nu}(x\ u_1\ ...\ u_n) = (x\ u_1\ ...\ u_n)/ =_{\beta\eta E}$.

If $(M_\alpha)_\alpha \models t = u$ then $\tilde{\nu}(t) = \tilde{\nu}(u)$ so $t/ =_{\beta\eta E} = u/ =_{\beta\eta E}$ i.e. $t =_{\beta\eta E} u$.

Remark: The converse is obviously false. If E contains an equation $x = y$ where x and y are two variables of type ι and M is a non trivial model then there exists an assignment ν such that $\nu(x) \neq \nu(y)$.

Remark: The proposition is obviously false if t and u do not have type ι. Indeed consider a variable f of type $\iota \to \iota$ and a set E which contains the equations $(f\ t) = t$ for all the terms t of type ι, we have $M \models f = \lambda x : \iota.x$ but $f \neq_{\beta\eta E} \lambda x : \iota.x$.

Remark: The proposition is obviously false if Γ contains variables of order greater than two. Indeed consider a variable f of type $\iota \to \iota$ and F of type $(\iota \to \iota) \to \iota$ and a set E which contains the equations $(f\ t) = t$ for all the terms t of type ι, we have $M \models (F\ f) = (F\ \lambda x : \iota.x)$ but $(F\ f) \neq_{\beta\eta E} (F\ \lambda x : \iota.x)$.

5 Finite Models

Definition: A model is said to be *finite* if the set M_ι is finite.

We want to sharpen the completeness theorem of section 3 and build a finite model. As remarked in section 3, the completeness theorem fails for such a model, so our completeness requirement will be weaker. For each closed term t, we are going to construct a finite model M_t such that for each closed term u, $M_t \models t = u$ if and only if $t =_{\beta\eta} u$. We do not require the model M_t to be uniform over t.

5.1 A Remark on the Relation $=_{\beta\eta E}$

Proposition: Let $E = \{a_i = b_i\}$ be a set of equations such that for every i, a_i and b_i have type ι. Let t and u be two terms such that $t =_{\beta\eta E} u$. Then either the normal forms of t and u are identical or they both have a subterm in the set $T = \{a_i, b_i\}$.
Proof: By induction on the structure of the proof of $t =_{\beta\eta E} u$.

- If $t =_{\beta\eta} u$ then the normal forms of t and u are identical.
- If $t = (c\ a)$ and $u = (c\ b)$ then if x has an occurrence in the normal form of $(c\ x)$ then a is a subterm of the normal form of $(c\ a)$ and b is a subterm of the normal form of $(c\ b)$. Otherwise the terms $(c\ a)$ and $(c\ b)$ have the same normal form.
- if $u =_{\beta\eta E} t$ then by induction hypothesis, either the normal forms of u and t are identical of they both have a subterm in T.
- If $t =_{\beta\eta E} v$ and $v =_{\beta\eta E} u$ then call t', v', u' the normal forms of t, v, u. By induction hypothesis, either
  - $t' = v'$ and $v' = u'$, in this case $t' = u'$,
  - t' and v' have a subterm in T and $v' = u'$, in this case t' and u' have a subterm in T,
  - $t' = v'$ and v' and u' have a subterm in T, in this case t' and u' have a subterm in T,

- t' and v' have a subterm in T and v' and u' have a subterm in T, in this case t' and u' have a subterm in T.

Corollary: Let $E = \{a_i = b_i\}$ be a set of equations such that for every i, a_i and b_i have type ι. Let t and u be two terms such that no subterm of the normal form of t is an a_i or a b_i, then $t =_{\beta\eta E} u$ if and only if $t =_{\beta\eta} u$.

5.2 Finite Models for Terms of Order Lower than Three

Before giving the theorem in its full generality, we shall consider the simpler case in which the order of the type of t is lower than three.

Proposition: Let t be a closed term which type is of order at most three. There exists a finite model M_t such that $M_t \models t = u$ if and only if $t =_{\beta\eta} u$, and the number of elements of M_ι is computable in function of t.
Proof: Let $\alpha = \beta_1 \to \ldots \to \beta_n \to \iota$ be the type of t. Let Γ be the context $\Gamma = \{x_1 : \beta_1, \ldots, x_n : \beta_n\}$. The types of the variables of Γ are of order at most two. Let u be a closed term of type α, we have $t =_{\beta\eta} u$ if and only if $(t\ x_1\ \ldots\ x_n) =_{\beta\eta} (u\ x_1\ \ldots\ x_n)$. Let E be the set containing all the equations $a = b$ with a and b in Λ_ι^Γ and neither a nor b is a subterm of the normal form of $(t\ x_1\ \ldots\ x_n)$. Using the proposition above $(t\ x_1\ \ldots\ x_n) =_{\beta\eta} (u\ x_1\ \ldots\ x_n)$ if and only if $(t\ x_1\ \ldots\ x_n) =_{\beta\eta E} (u\ x_1\ \ldots\ x_n)$.

Let us consider the model M_t constructed at the section 4. Obviously if $t =_{\beta\eta} u$ then $M_t \models t = u$. Conversely if we have $M_t \models t = u$ then $M_t \models (t\ x_1\ \ldots\ x_n) = (u\ x_1\ \ldots\ x_n)$. So $(t\ x_1\ \ldots\ x_n) =_{\beta\eta E} (u\ x_1\ \ldots\ x_n)$, therefore $(t\ x_1\ \ldots\ x_n) =_{\beta\eta} (u\ x_1\ \ldots\ x_n)$ and $t =_{\beta\eta} u$.

The number of elements of M_ι is $1 + k$ where k is the number of distinct subterms of the normal form of $(t\ x_1\ \ldots\ x_n)$ of type ι, it is therefore a computable in function of t.

5.3 General Case

Definition: Length of a Term

Let $t = \lambda x_1 : \alpha_1 \ldots . \lambda x_n : \alpha_n.(x\ d_1\ \ldots\ d_n)$ be a normal η-long term, we define the *length* of t ($|t|$) by induction on the number of variables occurrences of t

by

$$|t| = 1 + max\{|\lambda x_1 : \alpha_1 .... \lambda x_n : \alpha_n . d_i|\}$$

Consider a closed term t of type α, we shall prove that we can find terms $w_1, ..., w_p$ of type $\alpha \to \iota$ which free variables are of order at most two and such that for each closed term u of type α, $t =_{\beta\eta} u$ if and only if for all i, $(w_i\ t) =_{\beta\eta} (w_i\ u)$. Then we will be able to conclude as above.

Without loss of generality, we can assume that t is normal η-long. We shall first construct a term w such that $(w\ t) \neq_{\beta\eta} (w\ u)$ for all the normal η-long closed terms u that do not have the same length as t. Then for each normal η-long closed term u such that $u \neq t$ and u has the same length as t we shall construct a term w such that $(w\ t) \neq_{\beta\eta} (w\ u)$. Since the number of normal η-long closed terms which have a given length and a given type is finite, this will give us a finite number of w_i.

Proposition: Let t be a normal η-long closed term of type α, there exists a term w of type $\alpha \to \iota$ such that for every normal η-long closed term u of type α, if $|t| \neq |u|$ then $(w\ t) \neq_{\beta\eta} (w\ u)$ and the free variables of w are of order at most two.

Proof: For each integer p consider a variable z_p of type $\iota \to ... \to \iota \to \iota$. We define by induction over the structure of α a term c_α. Let us write

$$\alpha = \beta_1 \to ... \to \beta_p \to \iota$$

and

$$\beta_1 = \gamma_1^1 \to ... \to \gamma_{q_1}^1 \to \iota$$

$$...$$

$$\beta_p = \gamma_1^p \to ... \to \gamma_{q_p}^p \to \iota$$

Let us define

$$c_\alpha = \lambda x_1 : \beta_1 .... \lambda x_p : \beta_p . (z_p\ (x_1\ c_{\gamma_1^1} ... c_{\gamma_{q_1}^1})\ ...\ (x_p\ c_{\gamma_1^p} ... c_{\gamma_{q_p}^p}))$$

Now consider a term

$$t = \lambda y_1 : \alpha_1 .... \lambda y_n : \alpha_n . (y_i\ d_1\ ...\ d_p)$$

Let us write

$$\alpha_i = \beta_1 \to ... \to \beta_p \to \iota$$

and

$$\beta_1 = \gamma_1^1 \to ... \to \gamma_{q_1}^1 \to \iota$$

$$...$$

$$\beta_p = \gamma_1^p \to ... \to \gamma_{q_p}^p \to \iota$$

Let us define

$$e_1 = \lambda y_1 : \alpha_1 .... \lambda y_n : \alpha_n . d_1$$

$$...$$

$$e_p = \lambda y_1 : \alpha_1 .... \lambda y_n : \alpha_n . d_p$$

We have

$$(t\ c_{\alpha_1}\ ...\ c_{\alpha_n}) =_{\beta\eta} (z_p\ (e_1\ c_{\alpha_1}\ ...\ c_{\alpha_n}\ c_{\gamma_1^1}\ ...\ c_{\gamma_{m_1}^1})\ ...\ (e_p\ c_{\alpha_1}\ ...\ c_{\alpha_n}\ c_{\gamma_1^p}\ ...\ c_{\gamma_{m_p}^p}))$$

By induction on the length of t, the length of the normal form of the term $(t\ c_{\alpha_1}\ ...\ c_{\alpha_n})$ is the length of t. So if $|t| \neq |u|$ then the length of the normal forms of the terms $(t\ c_{\alpha_1}\ ...\ c_{\alpha_n})$ and $(u\ c_{\alpha_1}\ ...\ c_{\alpha_n})$ are different and $(t\ c_{\alpha_1}\ ...\ c_{\alpha_n}) \neq_{\beta\eta} (u\ c_{\alpha_1}\ ...\ c_{\alpha_n})$.

We take $w = \lambda x : \alpha.(x\ c_{\alpha_1}\ ...\ c_{\alpha_n})$. If $|t| \neq |u|$ then $(w\ t) \neq_{\beta\eta} (w\ u)$.

Proposition: Let t and u be two distinct normal η-long closed terms of type $\alpha = \alpha_1 \to ... \to \alpha_n \to \iota$. There are terms $c_1, ..., c_n$ of type $\alpha_1, ..., \alpha_n$ such that $(t\ c_1\ ...\ c_n) \neq_{\beta\eta} (u\ c_1\ ...\ c_n)$ and the free variables of $c_1, ..., c_n$ are of order at most two.

Proof: By induction on the number of variables occurrences of t. Let us write

$$t = \lambda x_1 : \alpha_1 .... \lambda x_n : \alpha_n .(x_i\ d_1\ ...\ d_p)$$

and

$$u = \lambda x_1 : \alpha_1 .... \lambda x_n : \alpha_n .(x_{i'}\ d'_1\ ...\ d'_{p'})$$

First Case: $i \neq i'$

Let $\alpha_i = \beta_1 \to ... \to \beta_p \to \iota$ and $\alpha_{i'} = \beta'_1 \to ... \to \beta'_{p'} \to \iota$. We let z and z' be two new variables of type ι and we take $c_i = \lambda y_1 : \beta_1 .... \lambda y_p : \beta_p . z$ and $c_{i'} = \lambda y_1 : \beta'_1 .... \lambda y_{p'} : \beta'_{p'} . z'$ and c_j be any term when $j \neq i$ and $j \neq i'$. We have $(t\ c_1\ ...\ c_n) =_{\beta\eta} z$ and $(u\ c_1\ ...\ c_n) =_{\beta\eta} z'$ so $(t\ c_1\ ...\ c_n) \neq_{\beta\eta} (u\ c_1\ ...\ c_n)$.

Second Case: $i = i'$

So we have $p = p'$. For all j we let $e_j = \lambda x_1 : \alpha_1 .... \lambda x_n : \alpha_n . d_j$ and $e'_j = \lambda x_1 : \alpha_1 .... \lambda x_n : \alpha_n . d'_j$. We have

$$t =_{\beta\eta} \lambda x_1 : \alpha_1 .... \lambda x_n : \alpha_n . (x_i \ (e_1 \ x_1 \ ... \ x_n) \ ... \ (e_p \ x_1 \ ... \ x_n))$$

and

$$u =_{\beta\eta} \lambda x_1 : \alpha_1 .... \lambda x_n : \alpha_n . (x_i \ (e'_1 \ x_1 \ ... \ x_n) \ ... \ (e'_p \ x_1 \ ... \ x_n))$$

Since $t \neq_{\beta\eta} u$ there exists an integer k such that $e_k \neq_{\beta\eta} e'_k$. Let us write $\alpha_i = \beta_1 \rightarrow ... \rightarrow \beta_p \rightarrow \iota$ and $\beta_k = \gamma_1 \rightarrow ... \rightarrow \gamma_m \rightarrow \iota$. The type of the terms e_k and e'_k is $\alpha_1 \rightarrow ... \rightarrow \alpha_n \rightarrow \gamma_1 \rightarrow ... \rightarrow \gamma_m \rightarrow \iota$. By induction hypothesis there exists terms $a_1, ..., a_n, b_1, ..., b_m$ such that

$$(e_k \ a_1 \ ... \ a_n \ b_1 \ ... \ b_m) \neq_{\beta\eta} (e'_k \ a_1 \ ... \ a_n \ b_1 \ ... \ b_m)$$

We let z be a new variable of type $\iota \rightarrow \iota \rightarrow \iota$. We let

$$c_i = \lambda y_1 : \beta_1 .... \lambda y_p : \beta_p . (z \ (y_k \ b_1 \ ... \ b_m) \ (a_i \ y_1 \ ... \ y_p))$$

and $c_j = a_j$ for every $j \neq i$.

Remark that for each j, $c_j[z \leftarrow \lambda x : \iota . \lambda y : \iota . y] =_{\beta\eta} a_j$, so

$$(e_k \ c_1 \ ... \ c_n \ b_1 \ ... \ b_m) \neq_{\beta\eta} (e'_k \ c_1 \ ... \ c_n \ b_1 \ ... \ b_m)$$

otherwise we would have $(e_k \ a_1 \ ... \ a_n \ b_1 \ ... \ b_m) =_{\beta\eta} (e'_k \ a_1 \ ... \ a_n \ b_1 \ ... \ b_m)$ by substituting the variable z by the term $\lambda x : \iota . \lambda y : \iota . y$. Now

$$(t \ c_1 \ ... \ c_n) =_{\beta\eta} (z \ (e_k \ c_1 \ ... \ c_n \ b_1 \ ... \ b_m) \ (a_i \ (e_1 \ c_1 \ ... \ c_n) \ ... \ (e_p \ c_1 \ ... \ c_n)))$$

and

$$(u \ c_1 \ ... \ c_n) =_{\beta\eta} (z \ (e'_k \ c_1 \ ... \ c_n \ b_1 \ ... \ b_m) \ (a_i \ (e'_1 \ c_1 \ ... \ c_n) \ ... \ (e'_p \ c_1 \ ... \ c_n)))$$

So

$$(t \ c_1 \ ... \ c_n) \neq_{\beta\eta} (u \ c_1 \ ... \ c_n)$$

Proposition: Let t and u be two distinct normal η-long closed terms of type α. there exists a term w of type $\alpha \rightarrow \iota$ such that $(w \ t) \neq_{\beta\eta} (w \ u)$ and the free variables of w are of order at most two.

Proof: Let $\alpha = \alpha_1 \to ... \to \alpha_n \to \iota$, by the previous proposition there are terms $c_1, ..., c_n$ such that $(t\ c_1\ ...\ c_n) \neq_{\beta\eta} (u\ c_1\ ...\ c_n)$, and the free variables of $c_1, ..., c_n$ are of order at most two. We take $w = \lambda x : \alpha.(x\ c_1\ ...\ c_n)$.

Proposition: Let t be a closed term t of type α, there exist terms $w_1, ..., w_p$ of type $\alpha \to \iota$ which free variables are of order at most two and such that for each term u, $t =_{\beta\eta} u$ if and only if for all i, $(w_i\ t) =_{\beta\eta} (w_i\ u)$.
Proof: We construct, using a previous proposition, a term w such that for each normal η-long closed term u of type α which length is different from the length of the normal η-long form of t, $(w\ t) \neq_{\beta\eta} (w\ u)$. Then for each normal η-long closed term u of type α which has the same length as the normal η-long form of t and which is different from the normal form of t, we construct, using a previous proposition, a term w such that $(w\ t) \neq_{\beta\eta} (w\ u)$. Since the number of normal η-long closed terms which have the same length and the same type as the normal η-long form of t is finite, this gives us a finite number of w_i.

Obviously, for each closed term u of type α, $t =_{\beta\eta} u$ if and only if for each integer i, $(w_i\ t) =_{\beta\eta} (w_i\ u)$.

Theorem: (Statman) Finite Completeness Theorem

Let t be a closed term of type α. There exists a finite model M_t such that $M_t \models t = u$ if and only if $t =_{\beta\eta} u$, and the number of elements of M_ι is computable in function of t.
Proof: Let $w_1, ..., w_n$ the terms given by the propsition above. Let E be the set containing all the equations $a = b$ with a and b in Λ_ι^Γ and neither a nor b is a subterm of the normal form of an $(w_i\ t)$ for some i. Using a proposition above $(w_i\ t) =_{\beta\eta} (w_i\ u)$ if and only if $(w_i\ t) =_{\beta\eta E} (w_i\ u)$.

Let us consider the model M_t constructed at the section 4. Obviously if $t =_{\beta\eta} u$ then we have $M_t \models t = u$. Conversely if $M_t \models t = u$ then $M_t \models (w_i\ t) = (w_i\ u)$. So $(w_i\ t) =_{\beta\eta E} (w_i\ u)$, therefore $(w_i\ t) =_{\beta\eta} (w_i\ u)$ and $t =_{\beta\eta} u$.

The number of elements of M_ι is $1 + k$ where k is the number of distinct subterms of the normal form of $(w_i\ t)$ of type ι, since the terms w_i are computable in function of t, the number of elements of M_ι is computable in function of t.

Corollary: Let t and u two terms of type α, $t =_{\beta\eta} u$ if and only if for all the finite standard models M, $M \models t = u$.

6 The λ-definability Conjecture

Because a simple function as the identity over integers is an infinite set (although it has a finite description), set-theoretical functions are not usually computational objects. In the same way, because completeness theorems concern usually infinite sets, model checking is not usually an effective decision procedure. Both argument fail when the sets involved are finite. Indeed, finite set-theoretical functions are computational objects (e.g. association lists) and finite model checking is an effective decision procedure (e.g. propositional calculus).

Definition: Let $(M_\alpha)_\alpha$ be the standard model with the base set $M = M_\iota$. A function f of M_α is said to be λ-definable if there exists a closed λ-term t of type α such that $\tilde{\nu}(t) = f$ (where ν is the only assignment over the empty set).

Conjecture: λ-definability Conjecture
If M is finite then it is decidable whether of not a function f of M_α is λ-definable.

Remark: The problem makes sense because M is finite, otherwise the functions of M_α would not be computational objects.

7 The λ-definability Conjecture Implies the Higher Order Matching Conjecture

Definition: Higher Order Matching Problem
A *higher order matching problem* is a pair of terms $< a, b >$ of types $\alpha_1 \rightarrow ... \rightarrow \alpha_n \rightarrow \beta$ and β. A solution to this problem is a n-uple of terms $< t_1, ..., t_n >$ of type $\alpha_1, ..., \alpha_n$ such that $(a\ t_1\ ...\ t_n) =_{\beta\eta} b$.

Conjecture : Higher Order Matching Conjecture
It is decidable whether of not a higher order matching problem has a solution.

Proposition: The higher order matching problem is decidable if and only if the higher order matching problem with closed a and b is decidable.

Proof: Let $< a, b >$ be a problem. Let $x_1 : \beta_1, ..., x_p : \beta_p$ be the free variables occurring in a and b. Let

$$\gamma_1 = \beta_1 \rightarrow ... \rightarrow \beta_p \rightarrow \alpha_1$$

$$...$$

$$\gamma_n = \beta_1 \rightarrow ... \rightarrow \beta_p \rightarrow \alpha_n$$

$$a' = \lambda y_1 : \gamma_1 .... \lambda y_n : \gamma_n . \lambda x_1 : \beta_1 .... \lambda x_p : \beta_p .(a\ (y_1\ x_1\ ...\ x_p)\ ...\ (y_n\ x_1\ ...\ x_p))$$

$$b' = \lambda x_1 : \beta_1 .... \lambda x_p : \beta_p . b$$

If we have

$$(a\ t_1\ ...\ t_n) = b$$

then

$$(a'\ \lambda x_1 : \beta_1 .... \lambda x_p : \beta_p . t_1\ ...\ \lambda x_1 : \beta_1 .... \lambda x_p : \beta_p . t_n) = b'$$

Conversely if we have

$$(a'\ t'_1\ ...\ t'_n) = b'$$

then

$$(a\ (t'_1\ x_1\ ...\ x_p)\ ...\ (t'_n\ x_1\ ...\ x_p)) = b$$

So the problem $< a, b >$ has a solution if and only if the problem $< a', b' >$ has one.

Proposition: The higher order matching problem is decidable if and only if the higher order matching problem with closed $t_1, ..., t_n$ is decidable.
Proof: Let $< a, b >$ be a problem. Let

$$a' = \lambda y_1 : \iota \rightarrow \alpha_1 .... \lambda y_n : \iota \rightarrow \alpha_n . \lambda x : \iota .(a\ (y_1\ x)\ ...\ (y_n\ x))$$

and

$$b' = \lambda x : \iota . b$$

(remark that if a and b are closed then a' and b' also are).

Assume the problem $< a, b >$ has a solution $t_1, ..., t_n$. Let z be a variable of type ι. For each type $\beta = \gamma_1 \rightarrow ... \rightarrow \gamma_k \rightarrow \iota$ we consider the term $w_\beta = \lambda z_1 : \gamma_1 .... \lambda z_k : \gamma_k . z$.

Let $x_1 : \beta_1, ..., x_p : \beta_p$ be the variables occurring free in the terms $t_1, ..., t_n$, $t'_i = t_i[x_1 \leftarrow w_{\beta_1}, ..., x_k \leftarrow w_{\beta_k}]$ and $u_i = \lambda z : \iota . t'_i$. We have

$$(a'\ u_1\ ...\ u_n) = b'$$

So the problem $< a', b' >$ has closed solution.

Now if the problem $< a', b' >$ has a closed solution $u_1, ..., u_n$ then let z be a variable of type ι and $t_i = (u_i\ z)$. We have

$$(a'\ u_1\ ...\ u_n) = b'$$

so

$$\lambda x : \iota.(a\ (u_1\ x)\ ...\ (u_n\ x)) = \lambda x : \iota.b$$

so

$$(a\ (u_1\ z)\ ...\ (u_n\ z)) = b$$

i.e.

$$(a\ t_1\ ...\ t_n) = b$$

So the problem $< a, b >$ has a solution if and only if the problem $< a', b' >$ has a closed solution.

Theorem: (Statman) If λ-definability is decidable then higher order matching is decidable.

Proof: Let us assume the λ-definability conjecture. We take two closed terms $a : \alpha_1 \rightarrow ...\alpha_n \rightarrow \beta$ and $b : \beta$ and consider the model M_b constructed in section 5. Let $A = \tilde{\nu}(a)$ and $B = \tilde{\nu}(b)$ (where ν is the only assignment over the empty set). By an enumeration procedure we select all the n-uples $< T_1, ..., T_n >$ such that $A(T_1, ..., T_n) = B$. The problem $< a, b >$ has a closed solution if and only if there is such a n-uple such that all the T_i are λ-definable.

Indeed if the problem $< a, b >$ has a closed solution $< t_1, ..., t_n >$, then let $T_i = \tilde{\nu}(t_i)$, the T_i are λ-definable and $A(T_1, ..., T_n) = B$. Conversely if there are λ-definable $T_1, ...T_n$ such that $A(T_1, ..., T_n) = B$. Let $t_1, ..., t_n$ be the terms such that $T_i = \tilde{\nu}(t_i)$. We have $M_b \models (a\ t_1\ ...\ t_n) = b$. So $(a\ t_1\ ...\ t_n) =_{\beta\eta} b$.

References

[1] H. Friedman, Equality Between Functionals, *Proceedings of Logical Colloquium 72-73*, Lecture Notes in Mathematics, 453, R. Parikh (Ed.), Springer-Verlag, 1975, pp. 22-37.

[2] L. Henkin, Completeness in the Theory of Types, *Journal of Symbolic Logic*, 15, 2, 1950, pp. 81-91.

[3] J.R. Hindley, J.P. Seldin, Introduction to Combinators and λ-Calculus, *Cambridge University Press*, 1986.

[4] G. Huet, Résolution d'Équations dans les Langages d'Ordre 1,2, ..., ω, *Thèse de Doctorat d'État*, Université de Paris 7, 1976.

[5] G. D. Plotkin, Lambda-definability and Logical Relations, *Memorandum SAI-RM-4*, University of Edinburgh, 1973.

[6] G. D. Plotkin, Lambda-Definability in the Full Type Hierarchy, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, J.R. Hindley and J.P. Seldin (Eds.), Academic Press, 1980, pp. 365-373.

[7] R. Statman, On the Existence of Closed Terms in the Typed λ-calculus II: Transformations of Unification Problems, *Theoretical Computer Science*, 15, 1981, pp. 329-338.

[8] R. Statman, Completeness, Invariance and λ-definability, *Journal of Symbolic Logic*, 47, 1, 1982, pp. 17-26.

[9] R. Statman, Equality Between Functionals Revisited, *Harvey Friedman's Research on the Foundations of Mathematics*, L. A. Harrington et al. (Ed.), North-Holland, 1985, pp. 331-338.

[10] D. A. Wolfram, The Clausal Theory of Types, *PhD Thesis*, Cambridge University, 1989.